LOCALIZATION PROPERTIES OF HIGHLY SINGULAR GENERALIZED FUNCTIONS

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We study the localization properties of generalized functions defined on a broad class of spaces of entire analytic test functions. This class, which includes all Gelfand–Shilov spaces $S^\beta_\alpha(\mathbb{R}^k)$ with $\beta < 1$, provides a convenient language for describing quantum fields with a highly singular infrared behavior. We show that the carrier cone notion, which replaces the support notion, can be correctly defined for the considered analytic functionals. In particular, we prove that each functional has a uniquely determined minimal carrier cone.

Keywords: generalized function, analytic functional, infrared singularity, carrier cone, plurisubharmonic function, Hörmander's L_2 estimates.

1. Introduction

In this paper, we study the localization properties of generalized functions defined on spaces of entire analytic test functions. The usual definition of the support of a generalized function is inapplicable in this case because of the lack of test functions with compact support (this difficulty is well known in the theory of hyperfunctions, where real-analytic test functions are used; see, e.g., Chap. 9 in [1]). The problem of finding a reasonable substitute for the support notion is important for extending the Wightman axiomatic approach to quantum gauge theory. Because of severe infrared singularities, gauge fields are generally well defined only under smearing with entire analytic functions in the momentum space (for example, this is the case for the Schwinger model in an arbitrary α-gauge [2]) and can therefore be treated neither in the original Wightman framework [3] using tempered distributions nor in a more general framework [4] based on Fourier hyperfunctions. This produces the problem of generalizing the spectral condition [5], whose standard formulation in terms of vacuum expectations relies heavily on the notion of the support of a generalized function

The localization properties of functionals defined on the Gelfand-Shilov spaces S^β_α with $\beta < 1$ were studied in [6, 7] (see [8] for the definition and properties of S^β_α; if $\beta < 1$, then S^β_α consists of entire analytic functions). It was shown that a carrier cone notion, which replaces the support notion, can be introduced consistently for such functionals. In particular, it was proved that each element of $S'^\beta_\alpha(\mathbb{R}^k)$ (the topological dual of $S^\beta_\alpha(\mathbb{R}^k)$) has a uniquely determined minimal carrier cone. Here, we extend the results in [6, 7] to a broader class of test function spaces previously used to analyze the spectral properties of sums of infinite series in Wick powers of indefinite-metric free fields [9]. This class is defined as follows.

Definition 1. Let $\alpha(s)$ and $\beta(s)$ be unbounded continuous monotonically increasing functions on the semiaxis $s \geq 0$. Let β be convex, and let there be a constant $\varkappa > 0$ such that the function $\alpha(s)/s^\varkappa$ is nondecreasing for sufficiently large s. For any $A, B > 0$, $\mathcal{E}^{\beta,B}_{\alpha,A}(\mathbb{R}^k)$ denotes the Banach space of all entire analytic functions on $\mathbb{C}^k$ with the finite norm

$$\sup_{z=x+iy\in\mathbb{C}^k} |f(z)|^{\alpha(|x/A|)-\beta(B|y|)}.$$

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The space $\mathcal{E}^\beta_\alpha(\mathbb{R}^k)$ is defined as the union $\bigcup_{A,B>0}\mathcal{E}^{\beta,B}_{\alpha,A}(\mathbb{R}^k)$ endowed with the inductive limit topology.

For definiteness, we everywhere assume that the norm $|\cdot|$ on $\mathbb{C}^k$ is uniform: $|z| = \max_{1\le j\le k}|z_j|$. For convex α, the spaces $\mathcal{E}^\beta_\alpha$ coincide with the spaces of type W described in Chap. 1 in [10]. If $\alpha(s) = s^{1/\mu}$ and $\beta(s) = s^{1/(\nu-1)}$, $\nu < 1$, then $\mathcal{E}^\beta_\alpha(\mathbb{R}^k) = S^\nu_\mu(\mathbb{R}^k)$ (to avoid confusion, we use S^ν_μ instead of the standard S^β_α). We call a cone W a conic neighborhood of a cone U if W has an open projection[1] and contains U. To define carrier cones, in addition to $\mathcal{E}^\beta_\alpha(\mathbb{R}^k)$, we introduce similar spaces associated with cones in $\mathbb{R}^k$.

Definition 2. Let U be a nonempty cone in $\mathbb{R}^k$ and α and β satisfy the conditions in Definition 1. For any $A, B > 0$, $\mathcal{E}^{\beta,B}_{\alpha,A}(U)$ denotes the Banach space of all entire analytic functions on $\mathbb{C}^k$ with the finite norm

$$\|f\|_{U,A,B} = \sup_{z\in\mathbb{C}^k} |f(z)|e^{-\rho_{U,A,B}(z)},$$

where

$$\rho_{U,A,B}(x+iy) = -\alpha(|x/A|) + \beta(B|y|) + \beta(B\delta_U(x)) \tag{1}$$

and $\delta_U(x) = \inf_{x'\in U}|x-x'|$ is the distance from x to U. The space $\mathcal{E}^\beta_\alpha(U)$ is defined by the relation $\mathcal{E}^\beta_\alpha(U) = \bigcup_{A,B>0,\,W\supset U}\mathcal{E}^{\beta,B}_{\alpha,A}(W)$, where W ranges all conic neighborhoods of U and the union is endowed with the inductive limit topology.

If $U = \mathbb{R}^k$, then Definition 2 is equivalent to Definition 1. Hereafter, we assume that all considered cones are nonempty. A closed cone K is called a carrier cone of a functional $u \in \mathcal{E}'^\beta_\alpha(\mathbb{R}^k)$ if u has a continuous extension to the space $\mathcal{E}^\beta_\alpha(K)$. Our main result in this paper is the following theorem.

Theorem 1. *Let the functions α and β satisfy the conditions in Definition 1. If the space $\mathcal{E}^\beta_\alpha(\mathbb{R}^k)$ is nontrivial (i.e., contains nonzero functions), then the following statements hold:*

1. *The space $\mathcal{E}^\beta_\alpha(\mathbb{R}^k)$ is dense in $\mathcal{E}^\beta_\alpha(U)$ for any cone $U \subset \mathbb{R}^k$.*
2. *If K_1 and K_2 are closed cones in $\mathbb{R}^k$, then for any $u \in \mathcal{E}'^\beta_\alpha(\mathbb{R}^k)$ carried by $K_1 \cup K_2$, there exist $u_{1,2} \in \mathcal{E}'^\beta_\alpha(\mathbb{R}^k)$ carried by $K_{1,2}$ such that $u = u_1 + u_2$.*
3. *If both K_1 and K_2 are carrier cones of $u \in \mathcal{E}'^\beta_\alpha(\mathbb{R}^k)$, then so is $K_1 \cap K_2$.*

We note that analogous results for Gelfand–Shilov spaces S^ν_μ were proved differently for $\nu = 0$ and $0 < \nu < 1$ in [6], [7]. Our approach here allows treating both these cases the same.

Statement 1 in Theorem 1 shows that the space of the functionals with the carrier cone K is naturally identified with the space $\mathcal{E}'^\beta_\alpha(K)$. By Definition 2, we have

$$\mathcal{E}^\beta_\alpha(K) = \bigcup_{W\supset K}\mathcal{E}^\beta_\alpha(W),$$

where the union is taken over all conic neighborhoods of K and is endowed with the inductive limit topology. It hence follows from Statement 1 in Theorem 1 that a functional $u \in \mathcal{E}'^\beta_\alpha(\mathbb{R}^k)$ is carried by K if and only if u has a continuous extension to the space $\mathcal{E}^\beta_\alpha(W)$ for every conic neighborhood W of K. Statement 3 in Theorem 1 implies that the intersection of an arbitrary family $\{K_\omega\}_{\omega\in\Omega}$ of carrier cones of a functional $u \in \mathcal{E}'^\beta_\alpha(\mathbb{R}^k)$ is again a carrier cone of u. Indeed, let W be a conic neighborhood of $K = \bigcap_{\omega\in\Omega}K_\omega$. Then by standard compactness arguments (cf. the proof of statement A in Lemma 9 below), there exists a finite

[1] By definition, the projection $\mathrm{Pr}\,W$ of a cone $W \subset \mathbb{R}^k$ is the image of $W \setminus \{0\}$ under the canonical map from $\mathbb{R}^k \setminus \{0\}$ to the sphere $\mathbb{S}_{k-1} = (\mathbb{R}^k \setminus \{0\})/\mathbb{R}_+$; the projection of W is assumed to be open in the topology of this sphere. We note that the degenerate cone $\{0\}$ is a cone with an open (empty) projection.

family $\omega_1, \ldots, \omega_n \in \Omega$ such that $\tilde{K} = \bigcap_{j=1}^{n} K_{\omega_j} \subset W$. By Statement 3 in Theorem 1, $\tilde{K}$ is a carrier cone of u, and u therefore has a continuous extension to $\mathcal{E}_{\alpha}^{\beta}(W)$. Hence, K is a carrier cone of u. In particular, each functional $u \in \mathcal{E}_{\alpha}'^{\beta}(\mathbb{R}^k)$ has a uniquely defined minimal carrier cone, the intersection of all carrier cones of u.

The proof of Theorem 1 essentially relies on using Hörmander's L_2 estimates for the solutions of the inhomogeneous Cauchy–Riemann equations[2] $\bar{\partial}_j \psi = \eta_j$, $j = 1, \ldots, k$. These estimates ensure the existence of a solution ψ that is square-integrable with respect to the weight function $e^{-\rho}/(1+|z|)^2$ if the η_j are square-integrable with respect to the weight function $e^{-\rho}$ and ρ is a plurisubharmonic function on $\mathbb{C}^k$. To illustrate how this result applies in our case, we briefly outline the proof of statement 1 in the theorem.

Let $\chi(z)$ be a smooth function on $\mathbb{C}^k$ vanishing for large $|z|$ and equal to unity in a neighborhood of the origin. For $f \in \mathcal{E}_{\alpha}^{\beta}(U)$, we construct an approximating sequence by setting $f_n(z) = f(z)\chi(z/n) - \psi_n(z)$, where the terms ψ_n are introduced to ensure the analyticity of f_n. This latter condition means that ψ_n satisfy the equations $\bar{\partial}_j \psi_n(z) = n^{-1} f(z)(\bar{\partial}_j \chi)(z/n)$. Hence, we can use the L_2 estimates to prove that the ψ_n can be chosen sufficiently small that $f_n \in \mathcal{E}_{\alpha}^{\beta}(\mathbb{R}^k)$ and $f_n \to f$ in $\mathcal{E}_{\alpha}^{\beta}(U)$. But this strategy implies using L_2-type norms, while $\mathcal{E}_{\alpha}^{\beta}(U)$ are defined by supremum norms. We resolve this problem in Sec. 2, where we derive an equivalent representation for $\mathcal{E}_{\alpha}^{\beta}(U)$ in terms of Hilbert spaces. Another complication is that the weight functions $e^{-\rho_{U,A,B}}$ in Definition 2 are not appropriate for L_2 estimates, because the functions $\rho_{U,A,B}$ are not plurisubharmonic. In Sec. 3, we overcome this difficulty by constructing suitable plurisubharmonic approximations for $\rho_{U,A,B}$. We prove Theorem 1 in Sec. 4.

2. Hilbert space representation for $\mathcal{E}_{\alpha}^{\beta}$

Let $A, B > 0$ and U be a cone in $\mathbb{R}^k$. We let $H_{\alpha,A}^{\beta,B}(U)$ denote the Hilbert space of all entire functions on $\mathbb{C}^k$ having the finite norm

$$\|f\|'_{U,A,B} = \left[\int |f(z)|^2 e^{-2\rho_{U,A,B}(z)} \, \mathrm{d}\lambda(z) \right]^{1/2}, \tag{2}$$

where $\mathrm{d}\lambda$ is the Lebesgue measure on $\mathbb{C}^k$ and $\rho_{U,A,B}$ is given by (1). We let $\tilde{\mathcal{E}}_{\alpha,A}^{\beta,B}(U)$ denote the space $\bigcap_{A'>A,\, B'>B} \mathcal{E}_{\alpha,A'}^{\beta,B'}(U)$ endowed with the topology defined by the norms $\|\cdot\|_{U,A',B'}$.

Lemma 1. *Let $A, B > 0$, U be a cone in $\mathbb{R}^k$, and α and β satisfy the conditions in Definition 1. Then $\tilde{\mathcal{E}}_{\alpha,A}^{\beta,B}(U)$ is a nuclear Fréchet space coinciding with $\bigcap_{A'>A,\, B'>B} H_{\alpha,A'}^{\beta,B'}(U)$ both as a set and topologically.*

Proof. The space $\tilde{\mathcal{E}}_{\alpha,A}^{\beta,B}(U)$ belongs to the class of the spaces $\mathcal{H}(M)$ introduced in [11]. The spaces $\mathcal{H}(M)$ and $\mathcal{H}_p(M)$ for $p \geq 1$ are defined[3] by a family $M = \{M_\gamma\}_{\gamma\in\Gamma}$ of strictly positive continuous functions on $\mathbb{C}^k$ and consist of all entire analytic functions on $\mathbb{C}^k$ having the respective finite norms

$$\sup_{z\in\mathbb{C}^k} M_\gamma(z)|f(z)|, \qquad \left[\int (M_\gamma(z))^p |f(z)|^p \, \mathrm{d}\lambda(z) \right]^{1/p}.$$

We suppose that (a) for any $\gamma_1, \gamma_2 \in \Gamma$, we can find $\gamma \in \Gamma$ and $C > 0$ such that $M_\gamma \geq C(M_{\gamma_1} + M_{\gamma_2})$ and (b) there exists a countable set $\Gamma' \subset \Gamma$ such that for every $\gamma \in \Gamma$, we can find $\gamma' \in \Gamma'$ and $C > 0$ such that $CM_\gamma \leq M_{\gamma'}$. Let $\Gamma = \{(A', B') : A' > A,\, B' > B\}$ and $M_{A',B'}(z) = e^{-\rho_{U,A',B'}(z)}$. Then all the above conditions are satisfied, and we have $\mathcal{H}(M) = \tilde{\mathcal{E}}_{\alpha,A}^{\beta,B}(U)$ and $\mathcal{H}_2(M) = \bigcap_{A',\,B'} H_{\alpha,A'}^{\beta,B'}(U)$. By Lemma 12 in [11], $\mathcal{H}(M)$ is a nuclear Fréchet space coinciding with $\mathcal{H}_p(M)$ for any $p \geq 1$ if the following conditions are satisfied:

[2]Here and hereafter, we use the short notation $\bar{\partial}_j$ for $\partial/\partial\bar{z}_j$.

[3]The definition of $\mathcal{H}(M)$ and $\mathcal{H}_p(M)$ given here is slightly less general than that in [11] but suffices for our purposes.

(I) For any $\gamma \in \Gamma$, there exists $\gamma' \in \Gamma$ such that $M_\gamma(z)/M_{\gamma'}(z)$ is integrable on $\mathbb{C}^k$ and tends to zero as $|z| \to \infty$.

(II) For any $\gamma \in \Gamma$, there exist $\gamma' \in \Gamma$, a neighborhood of the origin $\mathcal{B}$ in $\mathbb{C}^k$, and $C > 0$ such that $M_\gamma(z) \le C M_{\gamma'}(z+\zeta)$ for any $z \in \mathbb{C}^k$ and $\zeta \in \mathcal{B}$.

In the considered case, the satisfaction of conditions (I) and (II) respectively follows from Lemmas 2 and 3 below. The lemma is proved.

Lemma 2. *Let U be a cone in $\mathbb{R}^k$ and α and β satisfy the conditions in Definition 1. For any $A' > A > 0$ and $B' > B > 0$, $\sigma, \tau > 0$ can be found such that*

$$\rho_{U,A',B'}(z) - \rho_{U,A,B}(z) + C \ge \sigma |z|^\tau, \quad z \in \mathbb{C}^k, \tag{3}$$

where C is a constant and $\rho_{U,A,B}$ is given by (1).

Proof. Without loss of generality, we assume that $\beta(0) = 0$. Let $\varkappa$ satisfy the conditions in Definition 1, $s_0 > 0$ be such that $\alpha(s_0) > 0$, $\beta(s_0) > 0$, and the function $\mu(s) = \alpha(s)/s^\varkappa$ be nondecreasing for $s \ge s_0$. For $|x| \ge A' s_0$, we have

$$\alpha\left(\frac{|x|}{A}\right) - \alpha\left(\frac{|x|}{A'}\right) = \frac{|x|^\varkappa}{A^\varkappa}\mu\left(\frac{|x|}{A}\right) - \frac{|x|^\varkappa}{A'^\varkappa}\mu\left(\frac{|x|}{A'}\right) \ge$$
$$\ge \left(\frac{1}{A^\varkappa} - \frac{1}{A'^\varkappa}\right)\mu\left(\frac{|x|}{A}\right)|x|^\varkappa \ge \left(\frac{1}{A^\varkappa} - \frac{1}{A'^\varkappa}\right)\mu(s_0)|x|^\varkappa.$$

Because $\beta(0) = 0$, the convexity of β implies that $\beta(s) \le t\beta(s/t)$ for any $s \ge 0$ and $0 < t \le 1$. It hence follows that $\beta(s)/s$ is a nondecreasing function. We therefore have

$$\beta(B'|y|) - \beta(B|y|) \ge (B' - B)\frac{\beta(B'|y|)}{B'|y|}|y| \ge (B' - B)\frac{\beta(s_0)}{s_0}|y|$$

for $|y| \ge s_0/B'$. Setting $\tau = \min(1, \varkappa)$ and summing the estimates for α and β, we find that inequality (3) with $C = 0$ holds for large $|z|$ if σ is sufficiently small. Because all considered functions are continuous, adding a sufficiently large positive constant to the left-hand side ensures that the required bound holds for all $z \in \mathbb{C}^k$. The lemma is proved.

Lemma 3. *Let $R > 0$, U be a cone in $\mathbb{R}^k$, α and β be nondecreasing functions on $[0, \infty)$, and $\rho_{U,A,B}$ be given by (1). For any $A' > A > 0$ and $B' > B > 0$, there exists a constant C such that*

$$\rho_{U,A,B}(z + \zeta) \le \rho_{U,A',B'}(z) + C, \quad z, \zeta \in \mathbb{C}^k, \quad |\zeta| \le R.$$

Proof. Without loss of generality, we can assume that α and β are nonnegative. It then follows from the monotonicity of α and β that $\alpha((s+R)/A') \le \alpha(s/A) + \alpha(R/(A'-A))$ and $\beta(B(s+R)) \le \beta(B's) + \beta(RBB'/(B'-B))$. Let $z = x + iy$ and $\zeta = \xi + i\eta$ be such that $|\zeta| \le R$. Because $\delta_U(x+\xi) \le \delta_U(x) + |\xi|$, we have

$$\alpha\left(\frac{|x|}{A'}\right) \le \alpha\left(\frac{|x+\xi| + R}{A'}\right) \le \alpha\left(\frac{|x+\xi|}{A}\right) + \alpha\left(\frac{R}{A'-A}\right),$$
$$\beta(B|y+\eta|) + \beta(B\delta_U(x+\xi)) \le \beta(B'|y|) + \beta(B'\delta_U(x)) + 2\beta\left(\frac{RBB'}{B'-B}\right).$$

Summing these inequalities yields the required estimate.

Corollary 1. *If $f \in \mathcal{E}^{\beta}_{\alpha}(U)$, then $f(\cdot + \zeta) \in \mathcal{E}^{\beta}_{\alpha}(U)$ for any $\zeta \in \mathbb{C}^k$.*

We recall that dual Fréchet–Schwartz (DFS) spaces are by definition the inductive limits of sequences of locally convex spaces with injective compact linking maps (see [12]).

Lemma 4. *Let U be a cone in $\mathbb{R}^k$ and α and β satisfy the conditions in Definition 1. Then $\mathcal{E}^{\beta}_{\alpha}(U)$ is a nuclear DFS space coinciding (both as a set and topologically) with the space*

$$\bigcup_{A,B>0,\, W\supset U} H^{\beta,B}_{\alpha,A}(W),$$

where W ranges all conic neighborhoods of U and the union is endowed with the inductive limit topology.

Proof. Let $A' > A > 0$ and $B' > B > 0$, and let $W \supset W'$ be conic neighborhoods of U. Then we have continuous inclusion maps $\mathcal{E}^{\beta,B}_{\alpha,A}(W) \to \tilde{\mathcal{E}}^{\beta,B}_{\alpha,A}(W) \to \mathcal{E}^{\beta,B'}_{\alpha,A'}(W')$. We therefore have

$$\mathcal{E}^{\beta}_{\alpha}(U) = \bigcup_{A,B>0,\, W\supset U} \tilde{\mathcal{E}}^{\beta,B}_{\alpha,A}(W). \tag{4}$$

Because countable inductive limits of nuclear spaces are nuclear (see, e.g., the corollary to Theorem III.7.4 in [13]), the nuclearity of $\mathcal{E}^{\beta}_{\alpha}(U)$ follows from Lemma 1. Because all continuous maps from nuclear spaces to Banach spaces are nuclear (Theorem III.7.2 in [13]), the inclusion map $\mathcal{E}^{\beta,B}_{\alpha,A}(W) \to \mathcal{E}^{\beta,B'}_{\alpha,A'}(W')$ is nuclear as a composition of a nuclear map and a continuous map. It hence follows that $\mathcal{E}^{\beta}_{\alpha}(U)$ is a DFS space because nuclear maps are compact (Corollary 1 to Theorem III.7.1 in [13]). By Lemma 1, we have continuous inclusions $H^{\beta,B}_{\alpha,A}(W) \to \tilde{\mathcal{E}}^{\beta,B}_{\alpha,A}(W) \to H^{\beta,B'}_{\alpha,A'}(W')$. In view of (4), this implies that $\mathcal{E}^{\beta}_{\alpha}(U) = \bigcup_{A,B>0,\, W\supset U} H^{\beta,B}_{\alpha,A}(W)$. The lemma is proved.

3. Plurisubharmonic approximations

We recall that the norm $|\cdot|$ is assumed to be uniform.

Theorem 2. *Let $A, B > 0$, U be a nonempty cone in $\mathbb{R}^k$, α be a continuous nondecreasing function on $[0,\infty)$, and β be a continuous convex nondecreasing function on $[0,\infty)$. If there exists an entire function φ on $\mathbb{C}$ which is not identically zero and satisfies the bound*

$$|\varphi(z)| \le e^{\beta(|B_0 y|)-\alpha(|x/A_0|)}, \quad z = x + iy \in \mathbb{C}, \tag{5}$$

for some $A_0, B_0 > 0$, then for any $R > 0$, there exists a plurisubharmonic function ρ_R on $\mathbb{C}^k$ such that

$$\begin{aligned} &\rho_R(z) \le \rho_{\mathbb{R}^k,A',B'}(z) + \beta(2BeR), \quad z = x + iy \in \mathbb{C}^k,\\ &\rho_R(z) \le \rho_{U,A',B'}(z), \quad z \in \mathbb{C}^k,\\ &\rho_R(z) \ge \rho_{U,A,B}(z) - H, \quad |x| \le R, \end{aligned} \tag{6}$$

where $\rho_{U,A,B}$ is given by (1), H is a constant independent of R, $A' = 2A$, and $B' = (2ek+1)B + 4kA_0B_0/A$. If α is concave, then we can set $A' = A$.

Corollary 2. *Under the conditions of Theorem 2, there exists a plurisubharmonic function ρ such that*

$$\rho_{U,A,B}(z) - H \le \rho(z) \le \rho_{U,A',B'}(z), \quad z \in \mathbb{C}^k,$$

where $\rho_{U,A,B}$ is given by (1), H is a constant, $A' = 2A$, and $B' = (2ek+1)B + 4kA_0B_0/A$. If α is concave, then we can set $A' = A$.

Proof. Let ρ_R satisfy the conditions in Theorem 2. Then the function $\rho(z) = \overline{\lim}_{z'\to z} \sup_{R>0} \rho_R(z')$ is plurisubharmonic (Sec. II.10.3 in [14]) and satisfies the required estimate.

To prove Theorem 2, we need two lemmas.

Lemma 5. *Let α and β be continuous nondecreasing functions on $[0,\infty)$, and let there exist an entire analytic function φ on $\mathbb{C}$ that is not identically zero and satisfies the bound*

$$|\varphi(z)| \le e^{\beta(|y|)-\alpha(|x|)}, \quad z = x+iy \in \mathbb{C}. \tag{7}$$

Then there exist a plurisubharmonic function ρ on $\mathbb{C}^k$ and a constant H such that

$$-\alpha(2|x|) - k\beta(4|y|) - H \le \rho(z) \le k\beta(4|y|) - \alpha(|x|), \quad z = x+iy \in \mathbb{C}^k. \tag{8}$$

If α is concave, then there exist a plurisubharmonic function ρ on $\mathbb{C}^k$ and a constant H such that

$$-\alpha(|x|) - k\beta(2|y|) - H \le \rho(z) \le k\beta(2|y|) - \alpha(|x|), \quad z = x+iy \in \mathbb{C}^k. \tag{9}$$

Proof. Without loss of generality, we can assume that $\alpha(0) = \beta(0) = 0$ and $\varphi(0) \ne 0$ (if φ has a zero of order n at $z = 0$, then we can replace φ with $\tilde\varphi(z) = C\varphi(z)/z^n$; the function $\tilde\varphi$ satisfies (7) for sufficiently small C). We set

$$\begin{aligned} \tilde\rho(z) &= \sup_{\zeta\in\mathbb{C}^k} \{\Phi(z-\zeta) + M(\zeta)\}, \\ M(\zeta) &= \inf_{z'=x'+iy'\in\mathbb{C}^k} \{-\Phi(z'-\zeta) + k\beta(4|y'|) - \alpha(|x'|)\}, \end{aligned} \tag{10}$$

where $\Phi(z) = \sum_{j=1}^k \log|\varphi(2z_j)|$. We obviously have $\tilde\rho(z) \le k\beta(4|y|) - \alpha(|x|)$. Because Φ is plurisubharmonic, $\rho(z) = \overline{\lim}_{z'\to z} \tilde\rho(z)$ is also a plurisubharmonic function (see Sec. II.10.3 in [14]). In view of the continuity of α and β, we have $\tilde\rho(z) \le \rho(z) \le k\beta(4|y|) - \alpha(|x|)$, and it remains to show that $\tilde\rho(z) \ge -\alpha(2|x|) - k\beta(4|y|) - H$. It follows from (7) that

$$-\Phi(z'-z) \ge \alpha(2|x'-x|) - k\beta(2|y'-y|), \quad \zeta = \xi + i\eta,$$

and setting $H = -\Phi(0) = -k\log|\varphi(0)|$, we obtain

$$\tilde\rho(z) \ge -H + M(z) \ge \inf_{x',y'\in\mathbb{R}^k} \{k\beta(4|y'|) - k\beta(2|y'-y|) + \alpha(2|x'-x|) - \alpha(|x'|)\} - H. \tag{11}$$

Because both α and β are nonnegative and monotonic, we have

$$\beta(2|y'|) - \beta(|y'-y|) \ge -\beta(2|y|), \quad \alpha(2|x'-x|) - \alpha(|x'|) \ge -\alpha(2|x|). \tag{12}$$

Substituting these inequalities in (11), we obtain the required lower estimate for $\tilde\rho$. Thus, (8) is proved.

Now let α be concave. We replace $\beta(4|y'|)$ with $\beta(2|y'|)$ in definition (10) of $M(\zeta)$ and modify $\Phi(z)$ by setting $\Phi(z) = \sum_{j=1}^k \log|\varphi(z_j)|$. Defining $\tilde\rho$ and ρ as above, we obtain $\tilde\rho(z) \le \rho(z) \le k\beta(2|y|) - \alpha(|x|)$. Proceeding as above, we obtain the estimate

$$\tilde\rho(z) \ge \inf_{x',y'\in\mathbb{R}^k} \{k\beta(2|y'|) - k\beta(|y'-y|) + \alpha(|x'-x|) - \alpha(|x'|)\} - H. \tag{13}$$

Because α is concave and $\alpha(0) = 0$, we have $\alpha(s+t) \le \alpha(s) + \alpha(t)$ for any $s, t \ge 0$. It hence follows that

$$\alpha(|x+x'|) \le \alpha(|x|) + \alpha(|x'|)$$

for any $x, x' \in \mathbb{R}^k$. Changing $x' \to x' - x$, we obtain $\alpha(|x'-x|) - \alpha(|x'|) \ge -\alpha(|x|)$. Substituting this estimate and the first of inequalities (12) in (13) yields $\tilde\rho(z) \ge -\alpha(|x|) - k\beta(2|y|) - H$, which completes the proof of (9).

Lemma 6. *Let U be a cone in $\mathbb{R}^k$. For any $R>0$, there exists a plurisubharmonic function σ_R on $\mathbb{C}^k$ such that*

$$\sigma_R(z) \le k|y| + R, \quad z = x+iy \in \mathbb{C}^k, \tag{14}$$

$$\sigma_R(z) \le k|y| + \delta_U(x), \quad z \in \mathbb{C}^k, \tag{15}$$

$$\sigma_R(z) \ge \delta_U\left(\frac{x}{e}\right), \quad |x| \le R, \tag{16}$$

where $\delta_U(x) = \inf_{x'\in U} |x-x'|$ is the distance from x to U.

Proof. For any $a>0$, we define the subharmonic function Θ_a on $\mathbb{C}$:

$$\Theta_a(z) = a \log\left|\frac{\sin(z/a)}{z/a}\right|.$$

This function satisfies the inequalities

$$\Theta_a(iy) \ge 0, \quad y \in \mathbb{R}, \tag{17}$$

$$\Theta_a(z) \le |y| - a\log^+\left(\frac{|x|}{a}\right), \quad z = x+iy \in \mathbb{C}, \tag{18}$$

where $\log^+(r) = \max(\log r, 0)$. Indeed, because

$$\Theta_a(iy) = a\log\left(\frac{\sinh(y/a)}{y/a}\right),$$

estimate (17) follows from the inequality $\sinh y/y \ge 1$, $y \in \mathbb{R}$. Further, it follows from the inequalities

$$|\sin z| \le e^{|y|}, \quad \left|\frac{\sin z}{z}\right| \le e^{|y|}, \quad z = x+iy \in \mathbb{C},$$

that

$$\left|\frac{\sin(z/a)}{z/a}\right| \le e^{|y/a|}\min(1, a/|x|).$$

Passing to the logarithms, we obtain (18). We now set

$$\tilde{\sigma}_R(z) = \sup_{a>0,\, \xi\in\mathbb{R}^k,\, |\xi|\le R} \{\Phi_a(z-\xi) + M_a(\xi)\},$$

$$M_a(\xi) = \inf_{z'=x'+iy'\in\mathbb{C}^k} \{-\Phi_a(z'-\xi) + k|y'| + \delta_U(x')\},$$

where $\Phi_a(z) = \sum_{j=1}^k \Theta_a(z_j)$. We obviously have $\tilde{\sigma}_R(z) \le k|y| + \delta_U(x)$. By inequality (18), $\Phi_a(z-\xi) \le k|y|$ and therefore $\tilde{\sigma}_R(z) \le k|y| + \sup_{a>0,\,|\xi|\le R} M_a(\xi)$. Because $\Phi_a(0) = 0$, it follows from the definition of M_a that $M_a(\xi) \le \delta_U(\xi)$. Hence, $\tilde{\sigma}_R(z) \le k|y| + R$. Because Φ_a are plurisubharmonic functions, $\sigma_R(z) = \overline{\lim}_{z'\to z}\tilde{\sigma}_R(z)$ is also a plurisubharmonic function, and it follows from the continuity of $\delta_U(x)$ and $|y|$ that σ_R satisfies (14) and (15). Estimate (17) implies that $\Phi_a(iy) \ge 0$, $y \in \mathbb{R}^k$. Therefore,

$$\tilde{\sigma}_R(z) \ge \sup_{a>0}(\Phi_a(iy) + M_a(x)) \ge \sup_{a>0} M_a(x), \quad |x| \le R. \tag{19}$$

Using the elementary inequalities $\sum_{j=1}^{k} \log^+ |x_j| \geq \log^+(|x|)$ and $\sum_{j=1}^{k} |y_j| \leq k|y|$, we obtain

$$M_a(x) \geq \inf_{x' \in \mathbb{R}^k} \left\{ a \log^+ \left(\frac{|x'|}{a} \right) + \delta_U(x + x') \right\} \tag{20}$$

from estimate (18). Estimating $\delta_U(x+x')$ from below by $\max(\delta_U(x) - |x'|, 0)$ and calculating the infimum with respect to x', we obtain $M_a(x) \geq a \log^+(\delta_U(x)/a)$. Let $\delta_U(x) > 0$ and $a_0 = \delta_U(x)/e$. In view of (19), we find that

$$\tilde{\sigma}_R(z) \geq M_{a_0}(x) \geq \delta_U(x)/e, \quad |x| \leq R. \tag{21}$$

If $\delta_U(x) = 0$ and $|x| \leq R$, then the estimate $\tilde{\sigma}_R(z) \geq \delta_U(x)/e$ also holds because in view of (19) and (20), we have $\tilde{\sigma}_R(z) \geq 0$. Hence, (16) follows because $\sigma_R \geq \tilde{\sigma}_R$. The lemma is proved.

Proof of Theorem 2. Without loss of generality, we assume that $\beta(0) = 0$. We set $\rho'_R(z) = \beta(Be\sigma_R(z))$, where σ_R is a plurisubharmonic function satisfying the conditions in Lemma 6. Because a composition of a nondecreasing convex function with a plurisubharmonic function is plurisubharmonic (Theorem 4.1.13 and Sec. 4.1 in [15]), ρ'_R is a plurisubharmonic function. Because β is monotonic, inequalities (14)-(16) imply the estimates

$$\begin{aligned} &\rho'_R(z) \leq \beta(2Bek|y|) + \beta(2BeR), \quad z = x + iy \in \mathbb{C}^k, \\ &\rho'_R(z) \leq \beta(2Bek|y|) + \beta(2Be\delta_U(x)), \quad z \in \mathbb{C}^k, \\ &\rho'_R(z) \geq \beta(\delta_U(Bx)), \quad |x| \leq R. \end{aligned} \tag{22}$$

By Lemma 5, there exist a plurisubharmonic function ρ'' and a constant H such that

$$-\alpha(|x/A|) - k\beta(D|y|) - H \leq \rho''(z) \leq k\beta(D|y|) - \alpha(|x/A'|), \tag{23}$$

where $A' = 2A$ and $D = 2A_0B_0/A$ ($A' = A$ if α is concave). We set $\rho_R(z) = \rho'_R(z) + \rho''(z) + k\beta(D|y|) + \beta(|By|)$. The function ρ_R is plurisubharmonic because $\beta(D|y|)$ and $\beta(|By|)$ are convex and are therefore plurisubharmonic functions. Estimates (6) with $B' = 2kD + (2ek+1)B$ easily follow from (22), (23), and the inequality

$$2k\beta(D|y|) + \beta(B|y|) + \beta(2Bek|y|) \leq \beta(B'|y|),$$

which follows from the convexity of β and the condition $\beta(0) = 0$.

4. Proof of Theorem 1

As above, we let $\mathrm{d}\lambda$ denote the Lebesgue measure on $\mathbb{C}^k$. The proof of Theorem 1 is based on the following statement, which is a particular case of Theorem 4.2.6 in [15].

Lemma 7. *Let ρ be a plurisubharmonic function on $\mathbb{C}^k$ and η_j, $j = 1, \ldots, k$, be locally square-integrable functions on $\mathbb{C}^k$. If*

$$\int |\eta_j(z)|^2 e^{-\rho(z)}\, \mathrm{d}\lambda(z) < \infty$$

for all j and η_j (as generalized functions) satisfy the compatibility conditions $\bar{\partial}_j \eta_l = \bar{\partial}_l \eta_j$, then the inhomogeneous Cauchy–Riemann equations $\bar{\partial}_j \psi = \eta_j$ have a locally square-integrable solution satisfying the estimate[4]

$$2 \int |\psi(z)|^2 e^{-\rho(z)} (1 + |z|^2)^{-2}\, \mathrm{d}\lambda(z) \leq k^2 \sum_{j=1}^{k} \int |\eta_j(z)|^2 e^{-\rho(z)}\, \mathrm{d}\lambda(z).$$

[4]The estimate in Lemma 7 differs from the estimate in [15] by the factor k^2 in the right-hand side, which appears because we use the uniform norm instead of the Euclidean norm used in [15].

Let ρ be a measurable locally bounded function on $\mathbb{C}^k$. We let $L_2(\mathbb{C}^k, e^{-\rho}\mathrm{d}\lambda)$ denote the Hilbert space of functions square-integrable with respect to the measure $e^{-\rho}\mathrm{d}\lambda$ and H_ρ denote the closed subspace of $L_2(\mathbb{C}^k, e^{-\rho}\mathrm{d}\lambda)$ consisting of entire analytic functions.

Lemma 8. *Let ρ_0, ρ, and ρ' be measurable locally bounded functions on $\mathbb{C}^k$ such that $\rho_0 \le \rho'$ and $\rho \le \rho'$. If there exists a plurisubharmonic function ρ_R for any $R > 0$ such that*

$$\rho_R(z) + 2\log(1+|z|^2) \le \rho'(z), \quad z \in \mathbb{C}^k, \tag{24}$$

$$\rho_R(z) + 2\log(1+|z|^2) \le \rho_0(z) + C_R, \quad z \in \mathbb{C}^k, \tag{25}$$

$$\rho_R(z) \ge \rho(z), \quad |z| \le R, \tag{26}$$

where C_R is a constant, then H_ρ is contained in the closure of H_{ρ_0} in $H_{\rho'}$.

Proof. Let $f \in H_\rho$ and χ be a smooth function on $\mathbb{C}^k$ such that $0 \le \chi \le 1$, $\chi(z) = 1$ for $|z| \le 1$, and $\chi(z) = 0$ for $|z| \ge 2$. We set $g_n(z) = f(z)\chi(z/n)$. Because $\bar\partial_j g_n(z) = n^{-1} f(z)(\bar\partial_j \chi)(z/n)$ vanishes for $|z| \ge 2n$, it follows from (26) that

$$\int |\bar\partial_j g_n(z)|^2 e^{-\rho_{2n}(z)}\,\mathrm{d}\lambda(z) \le \int |\bar\partial_j g_n(z)|^2 e^{-\rho(z)}\,\mathrm{d}\lambda(z) \le \frac{a}{n^2}\|f\|_\rho^2, \quad j = 1, 2, \ldots, k,$$

where $\|\cdot\|_\rho$ is the norm in $L_2(\mathbb{C}^k, e^{-\rho}\mathrm{d}\lambda)$ and $a = \sup_{z,j} |\bar\partial_j \chi(z)|^2$. By Lemma 7, there exists a locally square-integrable function ψ_n on $\mathbb{C}^k$ such that $\bar\partial_j \psi_n = \bar\partial_j g_n$ and

$$\int |\psi_n|^2 e^{-\rho_{2n}(z)}(1+|z|^2)^{-2}\,\mathrm{d}\lambda(z) \le \frac{k^3 a}{2n^2}\|f\|_\rho^2. \tag{27}$$

In view of (25), this implies that $\|\psi_n\|_{\rho_0} < \infty$. Further, we have $\bar\partial_j(g_n - \psi_n) = 0$, and $g_n - \psi_n$ therefore coincides almost everywhere with an entire analytic function f_n. Because g_n is a function with compact support, we have $\|g_n\|_{\rho_0} < \infty$ and hence $f_n \in H_{\rho_0}$. Because $\rho \le \rho'$, we have

$$\|f - g_n\|_{\rho'}^2 \le \int_{|z|\ge n} |f(z)|^2 e^{-\rho(z)}\,\mathrm{d}\lambda(z).$$

Hence, $g_n \to f$ in $L_2(\mathbb{C}^k, e^{-\rho'}\mathrm{d}\lambda)$. By (24) and (27), we have $\|\psi_n\|_{\rho'}^2 \le k^3 a\|f\|_\rho^2/(2n^2)$. Therefore, $f_n \to f$ in $H_{\rho'}$, and the lemma is proved.

Lemma 8 was proved differently in [7] under the additional assumption that the ρ_R are smooth. The simple proof given above is closer to the line of reasoning sketched in Sec. 5 in [6].

We note that the spaces $H^{\beta,B}_{\alpha,A}(U)$ considered in Sec. 2 coincide with H_ρ for $\rho = 2\rho_{U,A,B}$.

Proof of Theorem 1.

1. Let $f \in \mathcal{E}^\beta_\alpha(U)$. By Lemma 4, there exist $A, B > 0$ and a conic neighborhood W of U such that $f \in H^{\beta,B}_{\alpha,A}(W)$. In view of Corollary 1, the nontriviality of $\mathcal{E}^\beta_\alpha(\mathbb{R}^k)$ implies the existence of $A_0, B_0 > 0$, and $f_0 \in \mathcal{E}^{\beta,B_0}_{\alpha,A_0}(\mathbb{R}^k)$ such that $f_0(0) \neq 0$ and $\|f_0\|_{\mathbb{R}^k,A_0,B_0} \le 1$. Then the entire function $\varphi(z) = f_0(z, 0, \ldots, 0)$ on $\mathbb{C}$ is not identically zero and satisfies (5). It follows from Lemma 2 and Theorem 2 that the functions $\rho = 2\rho_{W,A,B} - H$, $\rho_0 = 2\rho_{\mathbb{R}^k,A',B'}$, and $\rho' = 2\rho_{W,A',B'}$ satisfy the conditions in Lemma 8 if $A' > 2A$, $B' > (2ek+1)B + 4kA_0B_0/A$, and the constant H is sufficiently large. By Lemma 8, there exists a sequence $f_n \in H^{\beta,B'}_{\alpha,A'}(\mathbb{R}^k)$ tending to f in $H^{\beta,B'}_{\alpha,A'}(W)$. By Lemma 4, the Hilbert topology of $H^{\beta,B'}_{\alpha,A'}(W)$ is stronger than the topology induced from $\mathcal{E}^\beta_\alpha(U)$. Hence, $f_n \to f$ in $\mathcal{E}^\beta_\alpha(U)$.

2. Let $l\colon \mathcal{E}^{\beta}_{\alpha}(K_1\cup K_2)\to\mathcal{E}^{\beta}_{\alpha}(K_1)\oplus\mathcal{E}^{\beta}_{\alpha}(K_2)$ and $m\colon \mathcal{E}^{\beta}_{\alpha}(K_1)\oplus\mathcal{E}^{\beta}_{\alpha}(K_2)\to\mathcal{E}^{\beta}_{\alpha}(K_1\cap K_2)$ be the continuous linear maps respectively taking f to (f,f) and (f_1,f_2) to f_1-f_2. The map l has a closed image because we have $\mathcal{E}^{\beta}_{\alpha}(K_1)\cap\mathcal{E}^{\beta}_{\alpha}(K_2)=\mathcal{E}^{\beta}_{\alpha}(K_1\cup K_2)$ by Definition 2, and therefore $\operatorname{Im} l=\operatorname{Ker} m$. In view of Lemma 4, this implies that the space $\operatorname{Im} l$ is a DFS space.[5] Let $u\in\mathcal{E}'^{\beta}_{\alpha}(\mathbb{R}^k)$ be a functional carried by $K_1\cup K_2$ and $\hat u$ be its continuous extension to $\mathcal{E}^{\beta}_{\alpha}(K_1\cup K_2)$. The linear functional $\hat u l^{-1}$ is continuous on $\operatorname{Im} l$ by the open map theorem (see Theorem IV.8.3 in [13]; it is applicable because DFS spaces as strong duals of reflexive Fréchet spaces are B-complete [12]); by the Hahn–Banach theorem, there exists a continuous extension v of this functional to the entire space $\mathcal{E}^{\beta}_{\alpha}(K_1)\oplus\mathcal{E}^{\beta}_{\alpha}(K_2)$. Let v_1 and v_2 be the respective restrictions of v to $\mathcal{E}^{\beta}_{\alpha}(K_1)$ and $\mathcal{E}^{\beta}_{\alpha}(K_2)$. Then for any $f\in\mathcal{E}^{\beta}_{\alpha}(K_1\cup K_2)$, we have $\hat u(f)=v(f,f)=v_1(f)+v_2(f)$. This means that $u=u_1+u_2$, where $u_{1,2}$ are the restrictions of $v_{1,2}$ to $\mathcal{E}^{\beta}_{\alpha}(\mathbb{R}^k)$. By construction, $u_{1,2}$ are carried by the cones $K_{1,2}$.

3. Let l and m be as defined above, $u\in\mathcal{E}'^{\beta}_{\alpha}(\mathbb{R}^k)$ be a functional carried by both K_1 and K_2, and $u_{1,2}$ be its continuous extensions to $\mathcal{E}'^{\beta}_{\alpha}(K_{1,2})$. If the map m is surjective, then the open map theorem implies that $\mathcal{E}^{\beta}_{\alpha}(K_1\cap K_2)$ is topologically isomorphic to the quotient space $(\mathcal{E}^{\beta}_{\alpha}(K_1)\oplus\mathcal{E}^{\beta}_{\alpha}(K_2))/\operatorname{Ker} m$. We define the continuous linear functional v on $\mathcal{E}^{\beta}_{\alpha}(K_1)\oplus\mathcal{E}^{\beta}_{\alpha}(K_2)$ by the relation $v(f_1,f_2)=u_1(f_1)-u_2(f_2)$. By statement 1 in the theorem, u_1 and u_2 coincide on $\mathcal{E}^{\beta}_{\alpha}(K_1\cup K_2)$, and therefore $\operatorname{Ker} v\supset\operatorname{Im} l$. Because $\operatorname{Ker} m=\operatorname{Im} l$, this inclusion implies the existence of a functional $\hat u\in\mathcal{E}'^{\beta}_{\alpha}(K_1\cap K_2)$ such that $v=\hat u m$. If $f_{1,2}\in\mathcal{E}^{\beta}_{\alpha}(K_{1,2})$, then we have $\hat u(f_1)=v(f_1,0)=u_1(f_1)$ and $\hat u(f_2)=v(0,-f_2)=u_2(f_2)$. Hence, $\hat u$ is a continuous extension of u to $\mathcal{E}^{\beta}_{\alpha}(K_1\cap K_2)$. Proving statement 3 thus reduces to proving that m is surjective. The latter is implied by the following result on the decomposition of test functions.

Theorem 3. *Let $\mathcal{E}^{\beta}_{\alpha}(\mathbb{R}^k)$ be nontrivial, K_1 and K_2 be closed cones in $\mathbb{R}^k$, and $f\in\mathcal{E}^{\beta}_{\alpha}(K_1\cap K_2)$. Then there exist $f_{1,2}\in\mathcal{E}^{\beta}_{\alpha}(K_{1,2})$ such that $f=f_1+f_2$.*

In the next lemma, we summarize some simple facts about cones in $\mathbb{R}^k$ needed for proving Theorem 3.

Lemma 9. *Let K_1 and K_2 be closed cones in $\mathbb{R}^k$.*

A. *For any conic neighborhood W of $K_1\cap K_2$, there exist conic neighborhoods $V_{1,2}$ of $K_{1,2}$ such that $\bar V_1\cap\bar V_2\subset W$ (the bar means closure).*

B. *If $K_1\cap K_2=\{0\}$, then there exists $\theta>0$ such that $\delta_{K_1}(x)\ge\theta|x|$ for any $x\in K_2$.*

Proof. A. We let $\mathcal{C}$ denote the set of all cones in $\mathbb{R}^k$ containing the origin. By assumption, K_1, K_2, and W belong to $\mathcal{C}$. It is easy to see that the map $U\to\operatorname{Pr} U$ is a bijection between $\mathcal{C}$ and the set of all subsets of the sphere $\mathbb{S}_{k-1}=(\mathbb{R}^k\setminus\{0\})/\mathbb{R}_+$. Let Q denote its inverse map. It can be easily verified that both Pr and Q preserve closures, unions, and intersections. Hence, the $\operatorname{Pr} K_{1,2}$ are closed, and we have $\operatorname{Pr} K_1\cap\operatorname{Pr} K_2\subset\operatorname{Pr} W$. Because $\mathbb{S}_{k-1}$ is compact, there exist open neighborhoods $O_{1,2}$ of $\operatorname{Pr} K_{1,2}$ in $\mathbb{S}_{k-1}$ such that $\bar O_1\cap\bar O_2\subset\operatorname{Pr} W$. We set $V_{1,2}=Q(O_{1,2})$. Then $\bar V_1\cap\bar V_2=Q(\bar O_1\cap\bar O_2)\subset Q(\operatorname{Pr} W)=W$.

B. Let $K_2\ne\{0\}$ (if $K_2=\{0\}$, then the statement holds for any $\theta>0$). We set $F=\{x\in\mathbb{R}^k: x\in K_2 \text{ and } |x|=1\}$ and $\theta=\inf_{x\in F}\delta_{K_1}(x)$. Because F is compact and $F\cap K_1=\varnothing$, we have $\theta>0$. It remains to note that $\delta_{K_1}(x)=|x|\delta_{K_1}(x/|x|)\ge\theta|x|$ for any nonzero $x\in K_2$.

Lemma 10. *Let $A,B>0$, and let U_1, U_2, and U be cones in $\mathbb{R}^k$ such that $\bar U_1\cap\bar U_2=\{0\}$. If $\mathcal{E}^{\beta}_{\alpha}(\mathbb{R}^k)$ is nontrivial, then for any $f\in H^{\beta,B}_{\alpha,A}(U)$, there exist $A',B'>0$ and $f_{1,2}\in H^{\beta,B'}_{\alpha,A'}(U\cup U_{1,2})$ such that $f=f_1+f_2$.*

[5] We recall that the direct sum of a finite family of DFS spaces and a closed subspace of a DFS space are again DFS spaces (see [12]).

Proof. There exist conic neighborhoods $V_{1,2}$ of $U_{1,2}$ and measurable cones $W_{1,2}$ such that

$$W_1 \cup W_2 = \mathbb{R}^k, \quad W_1 \cap W_2 = \{0\}, \quad \bar{V}_\nu \cap \bar{W}_\nu = \{0\}, \quad \nu = 1,2. \tag{28}$$

Indeed, applying statement A in Lemma 9 to the closed cones $\bar{U}_1$ and $\bar{U}_2$,[6] we find conic neighborhoods $V_{1,2}$ of $\bar{U}_{1,2}$ such that $\bar{V}_1 \cap \bar{V}_2 = \{0\}$. Applying statement A in Lemma 9 to $\bar{V}_{1,2}$ again, we see that there exists a conic neighborhood W_2 of $\bar{V}_1$ such that $\bar{V}_2 \cap \bar{W}_2 = \{0\}$. We set $W_1 = (\mathbb{R}^k \setminus W_2) \cup \{0\}$. Then the first two relations in (28) obviously hold, and we have $\bar{V}_1 \cap \bar{W}_1 = \bar{V}_1 \cap W_1 = \{0\}$ because W_1 is closed.

Let g_0 be a nonnegative smooth function on $\mathbb{R}^k$ such that $g_0(x) = 0$ for $|x| \geq 1$ and

$$\int_{\mathbb{R}^k} g_0(x)\,\mathrm{d}x = 1.$$

We define smooth functions g_1 and g_2 on $\mathbb{C}^k$ by the relations

$$g_\nu(x+iy) = \int_{W_\nu} g_0(x-\xi)\,\mathrm{d}\xi, \quad x, y \in \mathbb{R}^k, \ \nu = 1,2.$$

By (28), we have $g_1 + g_2 = 1$. Applying statement B in Lemma 9 to the closed cones $\bar{U}_\nu$ and $(\mathbb{R}^k \setminus V_\nu) \cup \{0\}$, we conclude that there exists $\theta \in (0,1)$ such that $\delta_{U_\nu}(x) \geq \theta|x|$ for $x \notin V_\nu$, $\nu = 1,2$. Because $\delta_U(x) \leq |x|$ for any $x \in \mathbb{R}^k$, we have

$$\delta_U(\theta x) \leq \min(\delta_U(x), \theta|x|) \leq \min(\delta_U(x), \delta_{U_\nu}(x)) = \delta_{U \cup U_\nu}(x), \quad x \notin V_\nu. \tag{29}$$

Let $\tilde{W}_\nu = \{x \in \mathbb{R}^k : \delta_{W_\nu}(x) \leq 1\}$, $\nu = 1,2$. It follows from (28) and statement B in Lemma 9 that there exists $\theta' > 0$ such that $\delta_{W_\nu}(x) \geq \theta'|x|$ for $x \in \bar{V}_\nu$, $\nu = 1,2$. Hence, $\delta_{W_\nu}(x) > 1$ for all $x \in V_\nu$ such that $|x| \geq 1/\theta'$, i.e., the sets $V_\nu \cap \tilde{W}_\nu$ are bounded in $\mathbb{R}^k$. In view of (29), this implies that

$$\delta_U(x) \leq \delta_{U \cup U_\nu}\left(\frac{x}{\theta}\right) + C, \quad x \in \tilde{W}_\nu, \ \nu = 1,2, \tag{30}$$

where C is a constant. It hence follows that

$$\delta_U(x) \leq \delta_{U \cup U_1 \cup U_2}\left(\frac{x}{\theta}\right) + C, \quad x \in \tilde{W}_1 \cap \tilde{W}_2. \tag{31}$$

Let $\tilde{f}_{1,2} = f g_{1,2}$. Because f is analytic, we have $\bar{\partial}_j \tilde{f}_1 = f \bar{\partial}_j g_1$, $j = 1, \ldots, k$. By the definition of g_ν, we have $\operatorname{supp} g_\nu \subset \tilde{W}_\nu$, $\nu = 1,2$. Because $g_1 + g_2 = 1$, this implies $\operatorname{supp} \bar{\partial}_j g_1 \subset \tilde{W}_1 \cap \tilde{W}_2$, and in view of (2), (30), and (31), we obtain

$$\|\tilde{f}_\nu\|'_{U \cup U_\nu, A, \tilde{B}} \leq \tilde{C}\|f\|'_{U,A,B}, \quad \|\bar{\partial}_j \tilde{f}_1\|'_{U \cup U_1 \cup U_2, A, \tilde{B}} \leq \tilde{C}\|f\|'_{U,A,B}, \quad \nu = 1,2, \tag{32}$$

where $j = 1, \ldots, k$, $\tilde{B} = B/\theta$, and $\tilde{C}$ is a positive constant. As shown in the proof of statement 1 in Theorem 1, the nontriviality of $\mathcal{E}^\beta_\alpha(\mathbb{R}^k)$ implies the existence of an entire function φ on $\mathbb{C}$ satisfying (5). By Lemma 2 and Corollary 2, there exist $A' \geq A$, $B' \geq \tilde{B}$, and a plurisubharmonic function ρ such that

$$\rho_{U \cup U_1 \cup U_2, A, \tilde{B}}(z) - H \leq \rho(z) \leq \rho_{U \cup U_1 \cup U_2, A', B'}(z) - \log(1+|z|^2), \quad z \in \mathbb{C}^k, \tag{33}$$

[6] We note that the degenerate cone $\{0\}$ is a conic neighborhood of itself.

where H is a constant. It follows from (32) and (33) that

$$\int |\bar{\partial}_j \tilde{f}_1(z)|^2 e^{-2\rho(z)}\,\mathrm{d}\lambda(z) < \infty.$$

By Lemma 7, the inhomogeneous Cauchy–Riemann equations $\bar{\partial}_j \psi = \bar{\partial}_j \tilde{f}_1$ have a locally square-integrable solution such that

$$\int |\psi(z)|^2 e^{-2\rho(z)} (1+|z|^2)^{-2}\,\mathrm{d}\lambda(z) < \infty. \tag{34}$$

We have $\bar{\partial}_j(\tilde{f}_1 - \psi) = \bar{\partial}_j(\tilde{f}_2 + \psi) = 0$; therefore, there exist entire analytic functions f_1 and f_2 that respectively coincide almost everywhere with $\tilde{f}_1 - \psi$ and $\tilde{f}_2 + \psi$. It follows from the second inequality in (33) and condition (34) that $\|\psi\|'_{U\cup U_1\cup U_2, A', B'} < \infty$. In view of (32), it follows that $f_\nu \in H^{\beta,B'}_{\alpha,A'}(U \cup U_\nu)$, $\nu = 1, 2$. To complete the proof, it remains to note that $f = f_1 + f_2$ because continuous functions coinciding almost everywhere are equal.

Proof of Theorem 3. By Lemma 4, there exist $A, B > 0$ and a conic neighborhood W of $K_1 \cap K_2$ such that $f \in H^{\beta,B}_{\alpha,A}(W)$. By statement A in Lemma 9, we can find conic neighborhoods $V_{1,2}$ of $K_{1,2}$ such that $\bar{V}_1 \cap \bar{V}_2 \subset W$. Because W has an open projection, the cone $V = (\mathbb{R}^k \setminus W) \cup \{0\}$ is closed. Applying Lemma 10 to the closed cones $U_{1,2} = \bar{V}_{1,2} \cap V$ (obviously, $U_1 \cap U_2 = \{0\}$), we find $A', B' > 0$ and $f_{1,2} \in H^{\beta,B'}_{\alpha,A'}(W \cup U_{1,2})$ such that $f = f_1 + f_2$. Because $W \cup U_{1,2} \supset V_{1,2}$, it follows from Lemma 4 that $f_{1,2} \in \mathcal{E}^\beta_\alpha(K_{1,2})$. This completes the proof of Theorem 3 and statement 3 in Theorem 1.

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REFERENCES

1. L. Hörmander, *The Analysis of Linear Partial Differential Operators*: I. *Distribution Theory and Fourier Analysis* (Grundlehren Math. Wiss., Vol. 256), Springer, Berlin (1983).
2. A. Z. Capri and R. Ferrari, *J. Math. Phys.*, **25**, 141 (1984).
3. R. F. Streater and A. S. Wightman, *PCT, Spin and Statistics, and All That*, Benjamin, New York, 1964.
4. E. Brüning and S. Nagamachi, *J. Math. Phys.*, **30**, 2340 (1989).
5. U. Moschella and F. Strocchi, *Lett. Math. Phys.*, **24**, 103 (1992).
6. M. A. Soloviev, *Lett. Math. Phys.*, **33**, 49 (1995).
7. M. A. Soloviev, *Comm. Math. Phys.*, **184**, 579 (1997).
8. I. M. Gel'fand and G. E. Shilov, *Generalized Functions* [in Russian], Vol. 2, *Spaces of Fundamental and Generalized Functions*, Fizmatgiz, Moscow (1958); English transl., Acad. Press, New York (1968).
9. A. G. Smirnov and M. A. Solov'ev, *Theor. Math. Phys.*, **125**, 1349 (2000).
10. I. M. Gel'fand and G. E. Shilov, *Generalized Functions* [in Russian], Vol. 3, *Some Questions in the Theory of Differential Equations*, Fizmatgiz, Moscow (1958); English transl.: Vol. 3, *Theory of Differential Equations*, Acad. Press, New York (1967).
11. A. G. Smirnov, "On topological tensor products of functional Fréchet and DF spaces," arXiv:math/0512611v1 [math.FA] (2005).
12. H. Komatsu, *J. Math. Soc. Japan*, **19**, 366 (1967).
13. H. H. Schaefer, *Topological Vector Spaces*, MacMillan, New York, 1966.
14. V. S. Vladimirov, *Methods of the Theory of Functions of Many Complex Variables* [in Russian], Nauka, Moscow (1964); English transl., MIT Press, Cambridge, Mass. (1966).
15. L. Hörmander, *Notions of Convexity* (Progr. Math., Vol. 127), Birkhäuser, Boston (1994).